

**Rayleigh–Taylor Growth Rates for Arbitrary Density Profiles  
Calculated with a Variational Method**

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# Rayleigh-Taylor growth rates for arbitrary density profiles calculated with a variational method

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**Abstract:** The Rayleigh-Taylor instability exists at the interface of two fluids with different densities when the acceleration is pointing from the light fluid to the heavy one. It produces important effects in inertial confinement fusion (ICF). In order to understand more about the Rayleigh-Taylor instability, it is helpful to calculate the rates at which the perturbations grow at the interface. Assuming that the fluids are incompressible, using a variational method gives approximate growth rates for different density profiles. Growth rates are obtained using this method in a Fortran program and are compared with exact results calculated by integrating the Rayleigh equation in another Fortran program. The variational method is shown to be effective in solving for linear growth rates of the Rayleigh-Taylor instability.

## I. INTRODUCTION:

If one looks up in a dictionary, he will find the word "instability" being defined as "the quality or condition of being erratic or undependable." The physical concept of the Rayleigh-Taylor instability is a kind of instability as well. It exists at the interface of two fluids with different densities when the acceleration is pointing from the light fluid to the heavy fluid. It exists in everyday life, as illustrated in the example between oil and water.

In Fig.1, we have oil, the light fluid, on top of water, the heavy fluid. The perturbations exist at the interface. They are under the condition in which gravity pulls

down, which is equivalent to the fluid being accelerated upward. In Fig.2, we still have

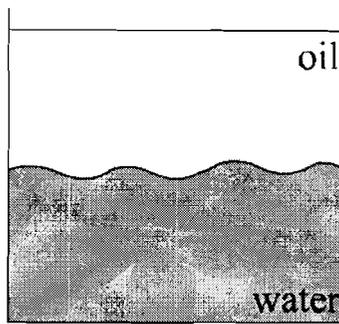


Fig.1

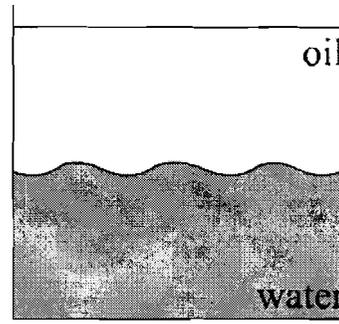


Fig.2

With gravity pulling down, which is equivalent to the fluid being accelerated upward, the state of oil on top of water is stable. The perturbations oscillate.

water on top of oil with perturbations existing. The condition does not change much, because this is a stable state. The perturbations oscillate.

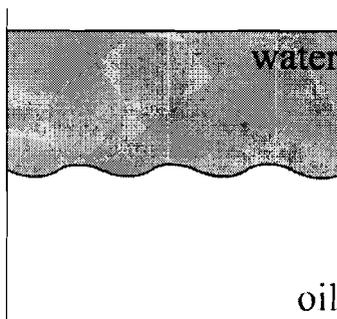


Fig.3

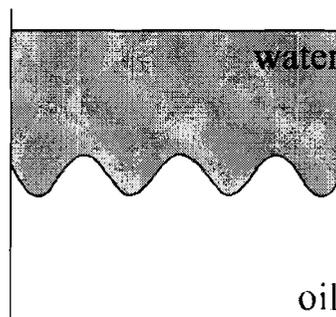


Fig.4

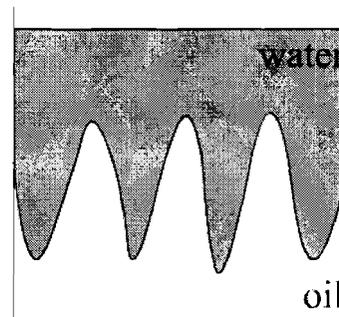


Fig.5

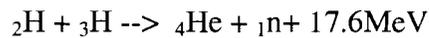
With gravity pulling down, meaning the acceleration going up, the state of water on top of oil is unstable. The perturbations grow exponentially with respect to time.

However, in Fig.3, water, the heavy fluid, is now on top of oil, the light fluid. With acceleration going up, it becomes unstable. The perturbations will not oscillate; rather, they will grow exponentially with respect to time. From Fig.3, Fig.4, to Fig.5, the perturbations have grown from little ripples to huge bubbles and spikes. Eventually, the perturbations will grow big enough that they will bring down the heavy fluid to exchange places with the light fluid. The water and oil will switch places and this unstable state will become a stable state as in Fig.1 and Fig.2 again. The unstable state described from Fig.3 to Fig.5 is called the Rayleigh-Taylor instability.

In the cases where I am going to solve for growth rates, the linear approximation is assumed to apply. The amplitudes of the perturbations are small compared to the wavelengths of the perturbations. In addition, ripples are assumed to vary sinusoidally across a planar interface.

## II. IMPORTANCE:

The Rayleigh-Taylor instability is important to Inertial Confinement Fusion. The fusion process is a nuclear reaction, which scientists hope to apply to the generation of energy. In this reaction, deuterium and tritium, two isotopes of hydrogen, react to produce one helium nucleus, one neutron, and 17.6MeV energy:



Two requirements for the fusion process are high temperature and high pressure. Scientists at the Laboratory for Laser Energetics shine laser beams onto a fuel target to help to create these two conditions.

Fig.6 shows a representative target. It consists of a thin plastic shell, a layer of deuterium and tritium ice fuel, and the core of deuterium and tritium gas fuel. When the laser begins to shine on the surface of the target, the plastic shell ablates. Vaporizing plastic accelerates outwardly, and this outward acceleration creates an inward acceleration, according to Newton's Third Law. In this acceleration phase, the interface becomes unstable due to the density difference between the ablated plastic shell and the plastic shell, with acceleration pointing from the expanding plastic vapor, the light fluid, to the solid plastic shell, the heavy fluid. The Rayleigh-Taylor instability exists here.

When the target is compressed small enough to create relatively high pressure inside the core, the target enters its deceleration phase. Reversing the direction of the acceleration, the interface between the ablated plastic shell vapor and the plastic shell becomes stable while the interface between the deuterium and tritium ice fuel and deuterium and tritium gas fuel becomes unstable. The Rayleigh-Taylor instability causes

the perturbations to grow at this interface. When the perturbations grow, they will bring the deuterium and tritium ice fuel, which is relatively cool, into the deuterium and tritium

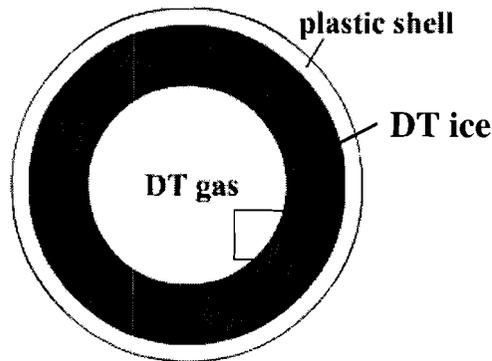


Fig.6: A target consists of a thin plastic shell, deuterium and tritium ice fuel, and deuterium and tritium gas fuel. During the acceleration phase, the Rayleigh-Taylor instability exists at the interface between the ablating plastic shell and the solid plastic shell. During the deceleration phase, it exists between the deuterium and tritium ice fuel and the deuterium and tritium gas fuel.

gas fuel to cool down the inner core. As a result, the high temperature requirement may not be met, and the fusion process may then not take place. In order to avoid this, scientists study the Rayleigh-Taylor instability, hoping to control the perturbations so that they will not grow big enough to ruin the fusion process. Thus, knowing the growth rates becomes important.

### III. SOLUTIONS:

#### 1. Exact Integration:

In order to solve for the exact growth rates, we use the Rayleigh equation:

$$d/dz(\rho dw/dz) - \rho k^2 w + (k^2 g/\gamma^2)(d\rho/dz)w = 0.$$

The Rayleigh equation is obtained from the physical theories of Newton's Second Law and the Conservation of Mass.<sup>1</sup> In addition, boundary conditions are imposed, i.e.  $dw/dz = \pm kw$ , which imposes exponentially decaying behavior at the left and right boundaries of the problem, respectively, where  $d\rho/dz = 0$ . This equation works for incompressible

fluids in a planar geometry. The variable  $z$  represents the position along the direction of acceleration, normal to the fluid interface, and  $\rho$  is the mass density at that point  $z$ . The constant  $k$  is the wave number; it equals to  $2\pi$  divided by the wavelength ( $\lambda$ ) of the perturbation.  $g$  is the acceleration of the fluids. The  $w(z)$  function and the parameter  $\gamma$  are the two things to be solved for.  $w(z)$  is the distribution of the  $z$  component of the perturbation velocity with respect to the  $z$  coordinate, and  $\gamma$  is the growth rate, which describes how fast the perturbations grow. For this equation, the assumed time dependence of the perturbation consists of a growth factor of  $e^{\gamma t}$ . The linear approximation applies to the perturbations to be solved for.

The Rayleigh equation is one example of a Liouville equation. According to the Sturm-Liouville Theory, for every eigenvalue, there will be an eigenfunction that satisfies the equation. In the Rayleigh equation,  $w(z)$  is the eigenfunction and the dimensionless quantity  $\Gamma = \gamma^2/kg$  is the eigenvalue. Another useful quantity in the mathematical manipulation is the dimensionless distance coordinate  $x = kz$ . Among all the possible eigenvalues and their corresponding eigenfunctions, my goal is to focus on the largest eigenvalue and the corresponding eigenfunction. That means there are many different possible perturbation velocity distributions in an unstable state, but the most important one is the one with the biggest growth rate. Since it grows the fastest, it has the biggest potential to destroy a fusion implosion. Therefore, we are looking for the largest value for  $\Gamma$ .

The shooting method is used in the exact integration for different density profiles. Boundary conditions are supplied for the problem. After guessing  $\Gamma$ ,  $w(z)$  is evaluated, and the eigenfunction is checked to see if it satisfies the boundary conditions. If they are not satisfied, the calculation is iterated, adjusting the eigenvalue and repeating the integration. This sequence is repeated until the boundary conditions are satisfied, indicating that the right eigenvalue and its corresponding eigenfunction have been found.

This exact integration of the Rayleigh equation works well for almost any density profile if the boundary conditions are provided. But if the boundary conditions cannot be easily defined, or if many iterations would be too time-consuming, what should

we do? In these situations, scientists will choose an approximate method over an exact integration.

## **2. The variational method:**

The variational method is a standard approximate method. The goal for my project is to test this method to see if it works well with several density profiles that are Rayleigh-Taylor unstable. For this method, we guess a trial function for the eigenfunction  $w(z)$  first. The trial function I used for  $w$  is  $w = e^{-|x-x_0|}$  as has been suggested by Mikaelian.<sup>2</sup> The density scale length is used here as an estimate of its center. The density scale length  $L$  equals  $\rho/(dp/dz)$ . A common rule is that this density scale length is a minimum where the eigenfunction will be centered. This rule is used to estimate  $x_0$ , the center of the trial eigenfunction. This function decays exponentially far from  $x = x_0$ , and no further boundary condition is imposed. If this trial function is a good choice, then *no* further iteration is required to obtain a good estimate of the growth rate. Once a proper trial function is chosen, there is no need to maximize or minimize any quantity. This is one very good thing about this implementation of the variational method. According to the "calculus of variations," the eigenvalue,  $\Gamma$ , is accurate to second order with respect to variations in the trial function about the exact eigenfunction. For the two density profiles that are shown below, the variational method works well. It provides good approximations.

## **IV. RESULTS:**

### **1. Ramp density profile:**

The ramp density profile jumps from one density to another over a spatial interval. The Atwood number is a quantity used to describe a density profile. It is the quotient of the difference between the two densities divided by the sum of these two densities. In the example shown in Fig.7, the ramp density profile jumps from 1 to 9 as  $x$

changes from -0.2 to 0.2. This gives us a profile with an Atwood number equal to 0.8 and a dimensionless ramp length equal to 0.4. Under the same boundary conditions as imposed before, the graphs are plotted from  $x = -3$  to 3.

In Fig.7, we show the graph obtained using the exact integration of the Rayleigh equation. The dashed line represents the density profile. The solid line is the eigenfunction,  $w(z)$ . The Fortran program generates the eigenvalue  $\Gamma = 0.754$  using the shooting method. In Fig.8, the graph is obtained using the variational method and the solid line is the trial function  $e^{-|x-x_0|}$ . The inverse of the scale length peaks where the density profile begins to jump, which is shown by the dotted line. The eigenfunction is centered at its peak, which is around  $x = -0.2$ .  $\Gamma$  is evaluated to be 0.753. This gives a -0.77% percent error in the eigenvalue. By moving the center of the eigenfunction  $x_0$ , it is found that the eigenvalue is not very sensitive to the changes. Fig.9 gives a comparison between the trial function and directly integrated result for the eigenfunction  $w(z)$ . The solid line is obtained by using the exact integration, and the dashed line is the trial function used in the variational method. The graph for the exact integration tends to have a round peak and the one for the variational method has a sharp peak. But overall, they are very close to each other.

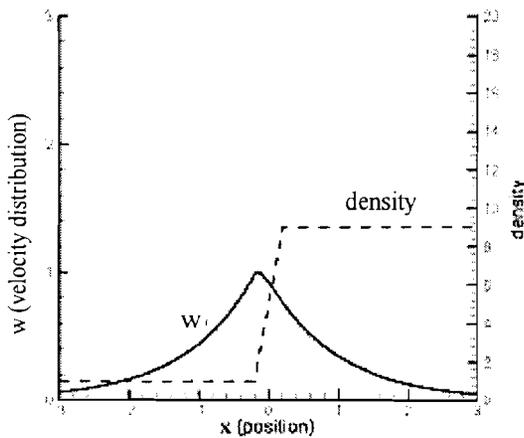


Fig.7: Exact integration for the ramp density profile.

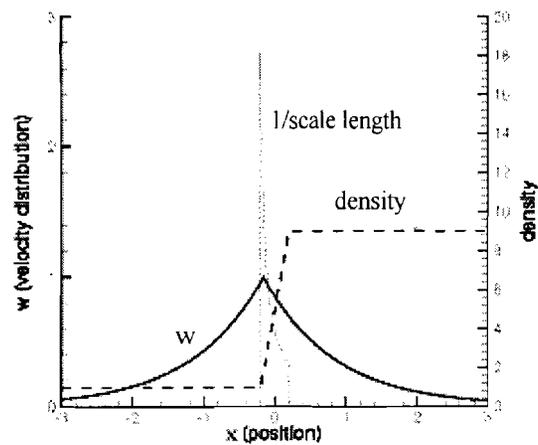


Fig.8: The variational method for the ramp density profile with one over scale length plotted to find the center of the eigenfunction.

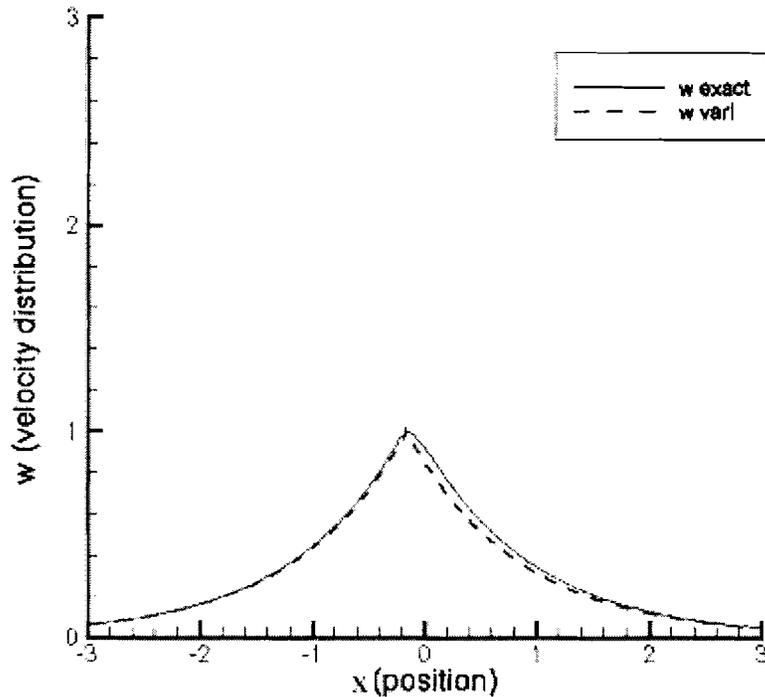


Fig. 9: Comparison between the real function and the trial function for the velocity distribution for the ramp density profile. The solid line is obtained by using the exact integration and the dashed line is the trial function used in the variational method. The graph for the exact integration tends to have a round peak and the one for the trial function has a sharp peak. But overall, they are very close to each other. The error in the eigenfunction yielded for this density profile is  $-0.77\%$ .

## 2. Exponentially growing density profile:

One other interesting density profile is the exponentially growing density profile. The density changes exponentially with respect to position within an interval. One thing that makes this density profile interesting is that the density scale length indication of the trial function center does not work because the density scale length is a constant over the interval where the density changes. So, the question comes up, where should we center the trial function? Should it be somewhere near where the density starts to jump, or in the center of the jumping interval, or at the end where the density stops

changing?

The density profile shown jumps from density one which is 1, to density two, which is 9, over the x interval -0.2 to 0.2. This gives a density profile with 0.8 Atwood number and 0.4 ramp length. The graphs are plotted from x = -3 to 3.

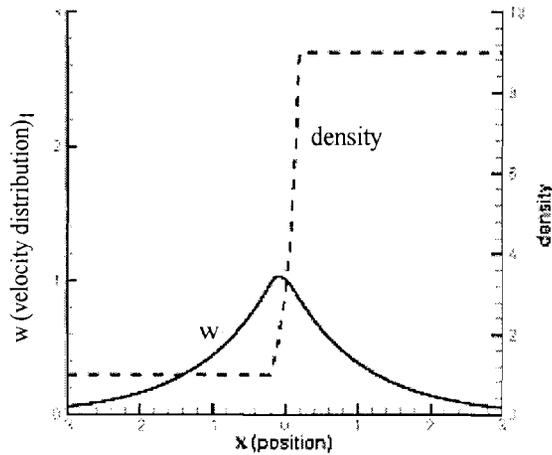


Fig.10: Exact integration for the exponentially growing density profile.

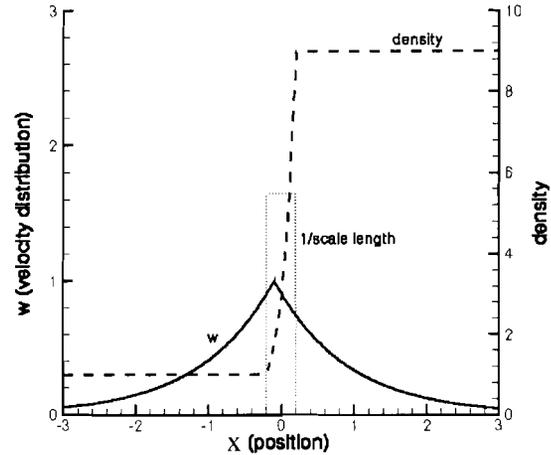


Fig.11: The variational method for the exponentially growing density profile showing the constant value of one over the density scale length over the entire width of the exponential density ramp.

In Fig.10, the dashed line represents the density profile. The solid line is the eigenfunction w. The program calculates the eigenvalue  $\Gamma = 0.750$  by iteration.

In Fig.11, for the same exponentially growing density profile shown as the dashed line, the solid line is the trial function  $e^{-|x-x_0|}$  for the eigenfunction. The interesting thing about this density profile is that the scale length is a constant over the interval

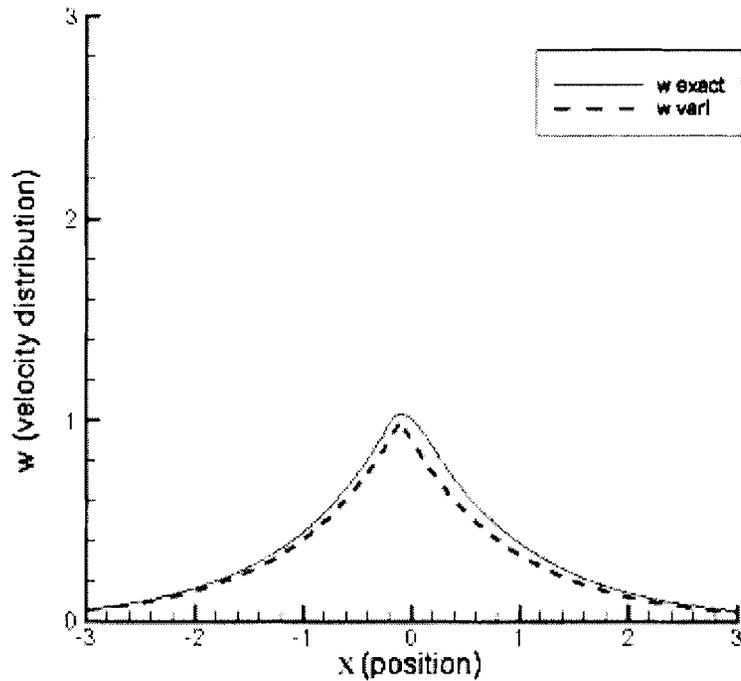


Fig.12: Comparison between the real function and the trial function for the velocity perturbation for the exponentially growing density profile. This density profile yields an error of  $-2.67\%$ .

where the density changes, so its minimum cannot be used as an indication of the eigenfunction center. As seen in the graph, the one-over-scale-length plot is a constant over the interval  $-0.2$  to  $0.2$ . By moving the center to get the maximum eigenvalue, it is shown that the center is near the middle of the density jump interval. So in this case, it is near the position  $x = 0$ . The approximate eigenvalue equals  $0.730$ . It gives an error of  $-2.67\%$  in the value of  $\Gamma$ . The eigenvalue is not very sensitive to the center of the eigenfunction. Fig.12 compares the two eigenfunctions, the result of the direct integration and the approximate trial function obtained from the variational method. The solid line is the exact solution and the dashed line is the approximate solution. The two curves are close to each other.

## V. ACCURACY:

### 1. Comparison between three approximate methods:

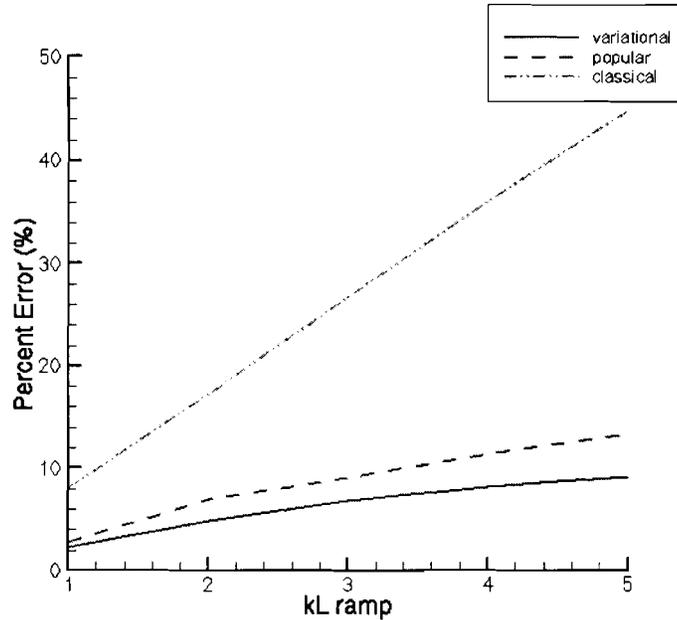


Fig. 13: The accuracy comparison between three different approximate methods is shown for the case of the linear density ramp with an Atwood number equal to 0.8 and a width  $L$ . The variational method has the least percent error, compared to the popular formula ( $\gamma^2 = Akg/(1+AkL)$ ) and the classical formula ( $\gamma^2 = Akg$ ).

There is not just one approximate method for calculating the Rayleigh-Taylor instability growth rate. One popular formula  $\gamma^2 = Akg/(1+AkL)$  is commonly used by scientists.<sup>3</sup> Another is the classical formula  $\gamma^2 = Akg$ . Fig.13 is generated for comparison among the variational method, the popular formula, and the classical formula by calculating the exact solution for each  $kL$ . The data are collected for the case of a linear density ramp with an Atwood number of 0.8 and a width  $L$ . The variational method is shown to be effective. The accuracy depends a lot on the density profile. When smaller Atwood numbers, such as 0.2, are used for a linear density ramp profile, the scale-length indication for the trial function center seems to not work as well. The best center of the eigenfunction does not occur at the minimum of the density scale length.

## 2. Growth factor and growth rate accuracy:

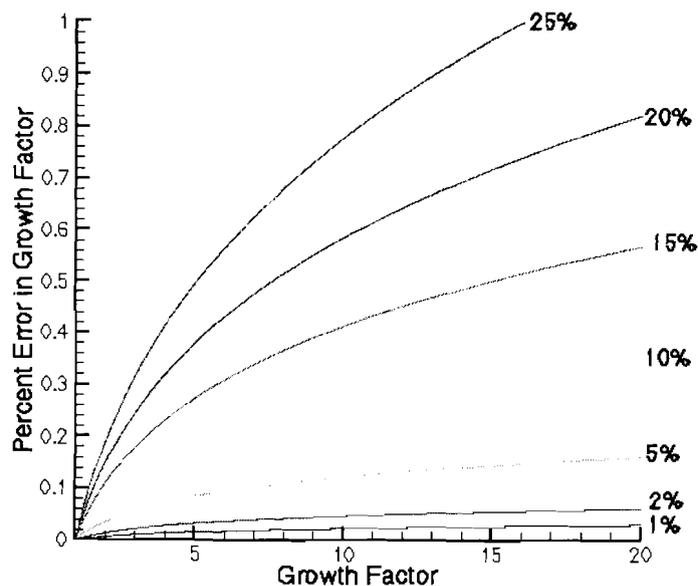


Fig.14: Growth factor accuracy is plotted for various values of the growth rate uncertainty, such as 1%, 2%, 5% and so on. The conversion between the growth rate accuracy and the growth factor accuracy can be obtained from the graph.

Fig.14 is a graph of the growth factor accuracy. It is plotted for various values of the growth rate uncertainty, such as 1%, 2%, 5%, and so on. The conversion between the growth factors and the growth rates is obtained through the following equation:

$$\text{Growth factor} = e^{\gamma t} \quad (\gamma \text{ being the growth rate})$$

One interesting growth factor for the NIF would be 10 and the accuracy of the growth factor should be 10%.<sup>4</sup> From the graph, which converts the accuracy in the growth factor into the accuracy in the growth rate, this point lies mostly near the 5% line. 5% accuracy in the growth rate is satisfied from the tests I did in my program using the variational method, so the variational method is shown to be acceptable for at least these cases.

## VI. CONCLUSION:

The results of the project indicate that the variational method is an acceptable method for approximating the growth rates for the Rayleigh-Taylor instability. While the percent error is largely dependent on the properties of the density profiles being solved for, Fig.13 shows that the variational method yields lower errors in the determination of the growth rate than either the popular or classical method for at least the case shown. For the linear ramp density profile, the trial eigenfunction centers on around where the density starts to change. For the exponentially growing density profile, the center is shown to be somewhere near the center of the density ramp. The indication of the eigenfunction center using the point of minimum scale length does not work well when the Atwood number is too small.

## VII. REFERENCES:

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2. K. O. Mikaelian, Phys. Rev. A **33**, 1216 (1985).
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4. V. Goncharov, private communication.

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